

# NONSQUEEZING PROPERTY OF THE COUPLED KdV TYPE SYSTEM WITHOUT MIURA TRANSFORM

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**ABSTRACT.** We prove the nonsqueezing property of the coupled Korteweg-de Vries (KdV) type system. Relying on Gromov's nonsqueezing theorem for finite dimensional Hamiltonian systems, the argument is to approximate the solutions to the original infinite dimensional Hamiltonian system by a frequency truncated finite dimensional system, and then the nonsqueezing property is transferred to the infinite dimensional system. This is the argument used by Bourgain [3] for the 1D cubic NLS flow, and Colliander et. al. [6] for the KdV flow. One of main ingredients of [6] is to use the Miura transform to change the KdV flow to the mKdV flow. In this work, we consider the coupled KdV system for which the Miura transform is not available. Instead of the Miura transform, we use the method of the normal form via the differentiation by parts. Although we present the proof for the coupled KdV flow, the same proof is applicable to the KdV flow, and so we provide an alternative simplified proof for the KdV flow.

## 1. INTRODUCTION

We consider a Hamiltonian dynamics property, the symplectic nonsqueezing, of the coupled KdV type system,

$$\begin{cases} u_t + u_{xxx} + \frac{1}{2}(uv)_x = 0 \\ v_t + v_{xxx} + (uv)_x = 0 \\ (u, v)|_{t=0} = (u_0(x), v_0(x)), \quad (u_0, v_0) \in H^{-1/2}(\mathbb{T}) \times H^{-1/2}(\mathbb{T}), \end{cases} \quad (\text{CKdV})$$

where  $(x, t) \in \mathbb{T} \times \mathbb{R} = [0, 2\pi) \times \mathbb{R}$ , and  $u, v$  are real valued functions. (CKdV) is a special version of the coupled KdV system,

$$\begin{cases} u_t + a_{11}u_{xxx} + a_{12}v_{xxx} + b_1uu_x + b_2uv_x + b_3u_xv + b_4vv_x = 0 \\ v_t + a_{21}u_{xxx} + a_{22}v_{xxx} + b_5uu_x + b_6uv_x + b_7u_xv + b_8vv_x = 0, \end{cases} \quad (1.1)$$

where  $u, v$  are real valued functions and  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is self-adjoint and non-singular. By diagonalization of  $A$ , we can reduce (1.1) to

$$\begin{cases} u_t + u_{xxx} + b_1uu_x + b_2uv_x + b_3u_xv + b_4vv_x = 0 \\ v_t + \alpha v_{xxx} + b_5uu_x + b_6uv_x + b_7u_xv + b_8vv_x = 0, \end{cases} \quad (1.2)$$

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where  $\alpha \neq 0$  on the same domain. There are many examples of this type of systems, such as the Majda-Biello system,

$$\begin{cases} u_t + u_{xxx} + \frac{1}{2}(uv)_x = 0 \\ v_t + \alpha v_{xxx} + (uv)_x = 0. \end{cases} \quad (1.3)$$

The KdV type equations can be seen as examples of nonlinear dispersive equations or Hamiltonian systems. If one consider them on a compact domain, due to lack of dispersion, it is better to see them as Hamiltonian systems. They have studied by many authors for both periodic and nonperiodic settings. The studies of the well-posedness have done via local smoothing estimates and Bourgain's  $X^{s,b}$  analysis [2, 12, 13]. In the low regularity below  $L^2$  the global well-posedness is obtained by I-method [4]. The Majda-Biello system (1.3) is an example of the coupled KdV system. As an extension of results of the KdV equations, the Cauchy problem of the Majda-Biello was well studied by Oh [18–20]. More precisely, the local well-posedness and the almost surely global well-posedness were proved in [19, 20], respectively. In [18], the global well-posedness of (1.3) in  $H^s(\mathbb{T}) \times H^s(\mathbb{T})$  for  $s \geq -1/2$ , when  $\alpha = 1$ , was proved (moreover, Oh [18] obtained the global well-posedness of (1.3) in  $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ ,  $s \geq \tilde{s}$ , where  $\tilde{s} = \tilde{s}(\alpha) \in (5/7, 1]$ , and  $\alpha$  satisfies the certain Diophantine condition). Note that the phase space should be  $H^{-1/2}(\mathbb{T}) \times H^{-1/2}(\mathbb{T})$  to consider the solution flow of (CKdV) as a symplectic map. Recently, Guo, Simon and Titi [9] proved the unconditional well-posedness of (CKdV) by the differentiation by parts.

The purpose of this paper is to show the symplectic nonsqueezing property of the solution flow of (CKdV). The Lebesgue measure is a typical invariant of a symplectic transform. In [8], Gromov discovered another invariant of a symplectic transform which is called Darboux width. Later, Hofer and Zehnder [11] developed the theory of the symplectic capacity. Moreover, Kuksin [14] extended the symplectic capacity to Hamiltonian PDEs. The main idea of [14] is that one can approximate the solution flow of the given Hamiltonian PDE as a finite dimensional symplectic map on the phase space. Concrete examples were studied by Bourgain [3] for the 1D cubic nonlinear Schrödinger equation (NLS), and Colliander et al. [6] for the KdV equation. Recently, Roumégoux [21] also proved nonsqueezing for the BBM equation. Mendelson [16] proved the nonsqueezing property of the Klein-Gordon equation on  $\mathbb{T}^3$  via a probabilistic approach.

As mentioned above, Bourgain [3] proved the nonsqueezing property of the 1D cubic NLS on  $L_x^2(\mathbb{T})$  space, and the basic strategy in [3] was an approximation by a finite dimensional truncated flow. The main step is to approximate the 1D cubic NLS flow which is the flow of the infinite dimensional Hamiltonian system by a frequency truncated finite dimensional system. Then due to Gromov's nonsqueezing of finite dimensional Hamiltonian systems, we have the nonsqueezing property of the truncated flow, and the result is transferred to the infinite dimensional NLS flow. Note that the truncated flow should be a symplectic map. Thus, the main here is to find a good such frequency truncation. The 1D cubic NLS is turned out to be well-behaved with the frequency truncations. Indeed, Bourgain used a basic (or a sharp) frequency truncation, and  $X^{s,b}$  space to apply this argument. Later, this argument extended by Colliander et al. [6] for the KdV flow on its phase space  $H_x^{-1/2}(\mathbb{T})$ . In [6], there are two additional ingredients. Firstly, it turned out that the sharp frequency truncation is

not working efficiently. They provided a counterexample that a sharp truncation does not approximate the original flow. Instead, they use a smooth truncation to resolve this problem. Secondly, they use the Miura transform to change the KdV flow to the mKdV flow. In fact, they proved the approximation by the truncated flow for the mKdV flow, and using the Miura transform and the inverse of it, concluded the approximation for the KdV flow.

The main goal of our work is to show the second ingredient, the Miura transform, is not necessary but can be replaced by more general technique, so called the normal form. In fact, the Miura transform is a special feature of the KdV flow due to integrability, and so it not widely applicable. Indeed, the system (CKdV) does not enjoy the Miura transform. Although we presented the proof for (CKdV), the same proof works for the KdV flow, and so we think this provides an alternative simplified proof of the result in [6].

The method of the normal form via the differentiation by parts first introduced by Babin, Ilyin and Titi [1] for the unconditional well-posedness of the KdV equation on  $L^2(\mathbb{T})$ , in which the normal form replaces the use of  $X^{s,b}$  spaces. This argument is extended to other equations [10, 15]. Also, Erdogan and Tzirakis [7] used this method with  $X^{s,b}$  multilinear estimates to show the global smoothing for the periodic KdV equation. The method of normal form is a way to detect and cancel out the nonresonancy in the nonlinear term. In general, if the characteristic surface is curved, then from the dispersion relation there is no quadratic resonance. Thus, by taking the normal form, the equation is changed to a cubic equation with quadratic boundary terms. See the detail in Section 4. In [6], we observe that the role of the Miura transform is to change the KdV equation to the mKdV equation to do analysis for trilinear nonlinearity. Thus, we have thought this could be replaced by the normal form method. Note that in this example, both the Miura transform (in [6]) and the method of normal form do not utilize full information of *integrability* of the KdV flow. Thus, the method of normal form is more widely applicable to nonintegrable equations.

The rest of paper is organized as follows: In Section 2, we present theorems for the non-squeezing property. In Section 3, we prove lemmas of bilinear and trilinear estimates in  $X^{s,b}$  space setting. In Section 4, we apply the differentiation by parts to the equation (CKdV) and prove key theorems using multilinear estimates.

### Notations.

For each dyadic number  $N$ , we denote the Littlewood-Paley projection by

$$\begin{aligned}\widehat{P_N u}(k) &:= 1_{N \leq |k| < 2N}(k) \hat{u}_k, \\ \widehat{P_{\leq N} u}(k) &:= 1_{|k| \leq N}(k) \hat{u}_k, \\ \widehat{P_{\geq N} u}(k) &:= 1_{|k| \geq N}(k) \hat{u}_k,\end{aligned}$$

where  $1_\Omega$  is a characteristic function on  $\Omega$ . For positive real numbers  $x, y$ ,  $x \lesssim y$  denotes  $x \leq Cy$  for some  $C > 0$ , and  $x \sim y$  means  $x \lesssim y$  and  $y \lesssim x$ . We also denote  $f = \mathcal{O}(g)$  by  $f \lesssim g$  for positive real valued functions  $f$  and  $g$ . Moreover,  $x \ll y$  denotes  $x \leq cy$  for some small positive constant  $c$ .

## 2. SETTING AND STATEMENT

We consider (CKdV) for simplicity of the argument. Denote  $S_{CKdV}(t)$  be the nonlinear solution flow of (CKdV). The system (CKdV) enjoys several conservation laws,

$$\begin{aligned} E_1 &= \int_{\mathbb{T}} u dx, \quad E_2 = \int_{\mathbb{T}} v dx, \\ M(u, v) &= \int_{\mathbb{T}} u^2 + v^2 dx, \end{aligned}$$

and

$$H(u, v) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 + v_x^2 - uv^2 dx. \quad (2.1)$$

Especially, (2.1) is the Hamiltonian, i.e., the system (CKdV) has Hamiltonian structure with respect to (2.1). We denote the spatial Fourier transform and the inverse Fourier transform by

$$\begin{aligned} \mathcal{F}_x(u) &= \hat{u}_k = \int_{\mathbb{T}} e^{-ikx} u(x) dx, \\ u(x) &= \int e^{ikx} \hat{u}(k) dk := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \hat{u}_k e^{ikx}. \end{aligned}$$

We use the spatial Sobolev space

$$\|u\|_{H_x^s} = \|\langle k \rangle^s \hat{u}\|_{L_k^2} := \frac{1}{(2\pi)^{1/2}} \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{u}|^2 \right)^{1/2},$$

where  $s \in \mathbb{R}$  and  $\langle k \rangle = (1 + |k|^2)^{1/2}$ . Mostly, we work on the mean zero  $H^s$  space as follows,

$$H_0^s = \left\{ u \in H^s : \frac{1}{2\pi} \int_{\mathbb{T}} u = 0 \right\} \quad \text{and} \quad \|u\|_{H_0^s} := \|\langle k \rangle^s \hat{u}(k)\|_{L_k^2}.$$

Since  $E_1$  and  $E_2$  are preserved quantities, the function space  $H_0^s \times H_0^s$  is well-suited for the solution to (CKdV). Note that due to the Galilean transform, one can switch from mean zero solutions to general mean solutions.

The equation (CKdV) is a Hamiltonian PDE associated with Hamiltonian (2.1). More precisely, we can write (CKdV) as

$$\begin{cases} u_t = \nabla_{\omega, u} H(u(t), v(t)) \\ v_t = \nabla_{\omega, v} H(u(t), v(t)), \end{cases}$$

where

$$\begin{cases} \omega(h, \nabla_{\omega, u} H(u(t), v(t))) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} H(u + \varepsilon h, v) \\ \omega(h, \nabla_{\omega, v} H(u(t), v(t))) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} H(u, v + \varepsilon h), \end{cases}$$

and

$$\omega(u, v) := \int_{\mathbb{T}} u \partial_x^{-1} v dx.$$

Thus, we say

$$\omega_H((u, u'), (v, v')) = \omega(u, v) + \omega(u', v') \quad (2.2)$$

the symplectic form associated with (2.1). Thus, a solution flow of (CKdV) is the Hamiltonian flow in  $(H_0^{-1/2}(\mathbb{T}) \times H_0^{-1/2}(\mathbb{T}), \omega_H)$  corresponding to (2.1). Note that the system is globally well-posed on its phase space  $H^{-1/2}(\mathbb{T}) \times H^{-1/2}(\mathbb{T})$ , and so the solution flow from data  $(u_0, v_0)$  to  $(u(t), v(t))$  is a symplectic map at any time  $t$ .

Now, we discuss the nonsqueezing theorem. We first recall Gromov's finite dimensional nonsqueezing theorem.

**Theorem 2.1** (Finite dimensional nonsqueezing theorem). *Let  $\mathcal{S}$  be the symplectic map on the  $2n$ -dimensional phase space. Let  $B_R, C_{k,r}$  denote a ball with radius  $R$ , and a cylinder with radius  $r$  at  $k$ -th component, respectively. If*

$$\mathcal{S}(B_R) \subseteq C_{k,r},$$

*then  $r \geq R$ .*

Our strategy is to find a frequency truncated finite dimensional solution flow which is also the Hamiltonian flow, and approximate to the original flow for some sense. Moreover, once we find the finite dimensional approximation, we can transfer the nonsqueezing theorem to the original flow.

The first guess is a sharp frequency truncation like [3], as an approximation of the flow. However, this is not a good approximation for (CKdV) (see Remark 2.3). Naturally, we next choose a smooth truncation like [6]. More precisely, let  $\phi(x)$  be a smooth even bump function supported to  $[-N, N]$  which equals 1 on  $[-N/2, N/2]$ , and  $b(k)$  be the restriction to integers of  $\phi(x)$ . We thus consider the smooth truncated system,

$$\begin{cases} \partial_t u + \partial_{xxx} u + \frac{1}{2} B((v)_x) = 0 \\ \partial_t v + \partial_{xxx} v + B((u)_x) = 0, \end{cases} \quad (\text{BKdV})$$

where

$$\widehat{Bu}(k) = b(k) \hat{u}(k).$$

Let  $S_{BKdV}(t)$  be the solution flow of (BKdV). Clearly,  $S_{BKdV}(t)$  is a finite dimensional solution flow. However,  $S_{BKdV}(t)$  is not a symplectic map, so we need more steps. To construct an appropriate finite dimensional symplectic map with respect to (2.2), we first consider a modified Hamiltonian. Let  $H_N(u, v)$  be a Hamiltonian which is defined by

$$H_N(u, v) := \frac{1}{2} \int u_x^2 + v_x^2 - B(u)(B(v))^2 dx,$$

on  $P_{\leq N} H_0^{-1/2}(\mathbb{T}) \times P_{\leq N} H_0^{-1/2}(\mathbb{T})$ . Then, we can get the appropriate truncated system by using  $H_N(u, v)$ . By the usual gradient with respect to (2.2),

$$\begin{aligned} \left. \frac{d}{d\varepsilon} H_N(u + \varepsilon w, v) \right|_{\varepsilon=0} &= \int \left( u_x w_x - B(w)(B(v))^2 \right) = \omega \left( w, -u_{xxx} - \frac{1}{2} B((B(v) B(v))_x) \right), \\ \left. \frac{d}{d\varepsilon} H_N(u, v + \varepsilon w) \right|_{\varepsilon=0} &= \int \left( v_x w_x - B(u) B(v) B(w) \right) = \omega \left( w, -v_{xxx} - B((B(u) B(v))_x) \right). \end{aligned}$$

Hence, we conclude that the smooth truncated system with respect to  $H_N(u, v)$  is given by

$$\begin{cases} \partial_t u + \partial_{xxx} u + \frac{1}{2} B((B(v) B(v))_x) = 0, \\ \partial_t v + \partial_{xxx} v + B((B(u) B(v))_x) = 0, \end{cases} \quad (\text{FKdV})$$

for initial data  $(u_0, v_0) \in P_{\leq N} H_0^{-1/2}(\mathbb{T}) \times P_{\leq N} H_0^{-1/2}(\mathbb{T})$ . Let  $S_{FKdV}(t)$  be the solution flow of (FKdV). It is the finite dimensional symplectic map at any time  $t$  by the construction and the global well-posedness. We now consider that  $S_{FKdV}(t)$  as a candidate of the good approximation.

**Remark 2.2.** We observe relation between (BKdV) and (FKdV). First of all, we apply the operator  $B$  to the both sides of (FKdV). Then we can obtain the system,

$$\begin{cases} \partial_t u + \partial_{xxx} u + \frac{1}{2} B^2((vv)_x) = 0 \\ \partial_t v + \partial_{xxx} v + B^2((uv)_x) = 0, \end{cases} \quad (\text{BBKdV})$$

for initial data  $(Bu_0, Bv_0) \in P_{\leq N} H_0^{-1/2}(\mathbb{T}) \times P_{\leq N} H_0^{-1/2}(\mathbb{T})$ . We let  $S_{BBKdV}(t)$  be the solution flow of (BBKdV), and then by the definition of  $S_{FKdV}(t)$  and  $S_{BBKdV}(t)$ ,

$$BS_{FKdV}(t)(u_0) = S_{BBKdV}(t)(Bu_0).$$

From the definition of  $B$ , (BBKdV) is (BKdV) with  $B$  replaced by  $B^2$ . This relation will be used in the proof of the approximation to the solution flow.

**Remark 2.3.** The sharp truncation (it uses  $P_{\leq N}$  instead of  $B$  for truncation) turns out to be not a good finite approximation, due to a counterexample by [6]. We consider the initial data

$$u_0 = v_0 = \sigma^3 \cos(k_0 x) + \sigma N^{1/2} \cos(Nx),$$

and by the similar iterating argument in [6] (or consecutive substitution), we can show that the sharp truncated coupled KdV flow does not approximate the original coupled KdV flow. Since the coupled KdV flow with the same initial data can be regarded as the KdV flow, the same counterexample as in [6] works in (CKdV).

**Remark 2.4.** By the same argument in [6, 18], I-method, we can show the global well-posedness of (FKdV). See [18] for the detail.

So far, we have chosen the appropriate truncation and the function spaces. We now define balls and cylinders in the phase space and state the main theorem, the nonsqueezing property of the coupled KdV type system (CKdV).

**Definition 2.5.** Let  $B_r^N(u_*)$  be a finite dimensional ball in  $P_{\leq N} H_0^{-1/2}$  which has radius  $r$  and centered at  $u_* \in P_{\leq N} H_0^{-1/2}$ . Likewise,  $B_r^\infty(u_*)$  is an infinite dimensional ball in  $H_0^{-1/2}$  which has radius  $r$  and centered at  $u_* \in H_0^{-1/2}$ . That is,

$$\begin{aligned} B_r^N(u_*) &:= \left\{ u \in P_{\leq N} H_0^{-1/2} : \|u - u_*\|_{H_0^{-1/2}} \leq r \right\}, \\ B_r^\infty(u_*) &:= \left\{ u \in H_0^{-1/2} : \|u - u_*\|_{H_0^{-1/2}} \leq r \right\}. \end{aligned}$$

For any  $k \in \mathbb{Z} \setminus \{0\} (:= \mathbb{Z}^*)$ , we define that  $C_{k,r}^N(z)$  is the finite dimensional cylinder in  $P_{\leq N}H_0^{-1/2}$  which has radius  $r$  and centered at  $z \in \mathbb{C}$ . Likewise,  $C_{k,r}^\infty(z)$  is the infinite dimensional cylinder in  $H_0^{-1/2}$  which has radius  $r$  and centered at  $z \in \mathbb{C}$ . That is,

$$C_{k,r}^N(z) := \left\{ u \in P_{\leq N}H_0^{-1/2} : |k|^{-1/2} |\hat{u}_k - z| \leq r \right\},$$

$$C_{k,r}^\infty(z) := \left\{ u \in H_0^{-1/2} : |k|^{-1/2} |\hat{u}_k - z| \leq r \right\}.$$

Now we state our main theorem.

**Theorem 2.6.** *Let  $k_1, k_2 \in \mathbb{Z}^*$ ,  $r_1 < R_1$ ,  $r_2 < R_2$  and  $T > 0$ . In addition,  $(u_*, v_*) \in H_0^{-\frac{1}{2}}(\mathbb{T}) \times H_0^{-\frac{1}{2}}(\mathbb{T})$  and  $(z, w) \in \mathbb{C}^2$ . Then*

$$S_{CKdV}(T) (B_{R_1}^\infty(u_*) \times B_{R_2}^\infty(v_*)) \not\subseteq C_{k_1, r_1}^\infty(z) \times C_{k_2, r_2}^\infty(w).$$

*In other words, there exists a global solution  $S_{CKdV}(t)(u_0, v_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$  to (CKdV) such that*

$$\|u_0 - u_*\|_{H_0^{-1/2}} \leq R_1, \quad |k_1|^{-1/2} |(S_{CKdV}(T)u_0)^\wedge(k_1) - z| > r_1,$$

*and*

$$\|v_0 - v_*\|_{H_0^{-1/2}} \leq R_2, \quad |k_2|^{-1/2} |(S_{CKdV}(T)v_0)^\wedge(k_2) - w| > r_2,$$

*respectively*<sup>1</sup>.

Note that no smallness conditions are imposed on  $k_i$ ,  $r_i$ ,  $R_i$ ,  $(u_*, v_*)$  and  $(z, w)$ . Our strategy is to construct a truncated solution flow which has the nonsqueezing property, and approximate to the original solution flow. Hence, we need the nonsqueezing theorem associated with the truncated solution flow (FKdV).

**Lemma 2.7.** *Let  $k_1, k_2 \in \mathbb{Z}^*$  such that  $|k_1|, |k_2| \leq N$ . Let  $r_1 < R_1$ ,  $r_2 < R_2$  and  $T > 0$ . Furthermore, let  $(u_0, v_0) \in P_{\leq N}H_0^{-\frac{1}{2}}(\mathbb{T}) \times P_{\leq N}H_0^{-\frac{1}{2}}(\mathbb{T})$  and  $z, w \in \mathbb{C}$ . Then*

$$S_{FKdV}(T) (B_{R_1}^N(u_0) \times B_{R_2}^N(v_0)) \not\subseteq C_{k_1, r_1}^N(z) \times C_{k_2, r_2}^N(w)$$

Since  $S_{FKdV}(T)$  is the finite dimensional symplectic map at time  $T$ , Lemma 2.7 is a direct consequence of Theorem 2.1. Thus, in the rest we prove that two flows, the flow of (CKdV) and (FKdV) are close for sufficiently large  $N$ . We show that in two steps. Firstly, we prove that solutions agreeing on low frequency data stay close at frequencies  $\leq N$ . Secondly, we show that solutions to the truncated flow stay close to the original flow in low frequencies. The first part is written as follows,

**Theorem 2.8.** *Let  $T > 0$ ,  $\varepsilon > 0$ ,  $(u_0, v_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$  and  $(u'_0, v'_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$ . There exists a positive integer*

$$N_0(T, \varepsilon, \|u_0\|_{H_0^{-1/2}}, \|u'_0\|_{H_0^{-1/2}}, \|v_0\|_{H_0^{-1/2}}, \|v'_0\|_{H_0^{-1/2}})$$

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<sup>1</sup>Obviously,  $S_{CKdV}(t)$  is the flow  $\mathbb{R} \rightarrow \mathbb{R}^2$ . However by abuse of notation, let  $S_{CKdV}(t)u_0$  and  $S_{CKdV}(t)v_0$  denote the first and the second component of  $S_{CKdV}(t)(u_0, v_0)$ , respectively. Here and in the sequel, we use these notations for all solution flow as well.

such that for all  $N > N_0$ , and the data satisfying  $P_{\leq 2N}(u_0, v_0) = P_{\leq 2N}(u'_0, v'_0)$ ,

$$\begin{aligned} & \sup_{|t| \leq T} \|P_{\leq N}(S_{CKdV}(t)u_0 - S_{CKdV}(t)u'_0)\|_{H_0^{-1/2}} \\ & + \sup_{|t| \leq T} \|P_{\leq N}(S_{CKdV}(t)v_0 - S_{CKdV}(t)v'_0)\|_{H_0^{-1/2}} \\ & \lesssim \varepsilon. \end{aligned}$$

We now compare the solutions to the original flow and the truncated flow. The proof of this case is more involved and form main analysis of this work. However, we introduce a relatively easier way than the former result [6]. In this step, we use the method of the normal form to change the flow with trilinear nonlinear terms and bilinear boundary terms. See the detail in Section 4.

**Theorem 2.9** (Truncation of the flow). *Let  $T > 0$  and  $\varepsilon > 0$ . There exists a positive integer  $N_0(T, \varepsilon, \|u_0\|_{H_0^{-1/2}}, \|v_0\|_{H_0^{-1/2}})$  such that for all  $N > N_0$ ,*

$$\begin{aligned} & \sup_{|t| \leq T} \|P_{\leq N^{\frac{1}{2}}}(S_{CKdV}(t)u_0 - S_{BKdV}(t)u_0)\|_{H_0^{-\frac{1}{2}}} \\ & + \sup_{|t| \leq T} \|P_{\leq N^{\frac{1}{2}}}(S_{CKdV}(t)v_0 - S_{BKdV}(t)v_0)\|_{H_0^{-\frac{1}{2}}} \\ & \lesssim \varepsilon, \end{aligned} \tag{2.3}$$

where  $(u_0, v_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$  which has the frequency support on  $[-N, N] \times [-N, N]$ .

Note that we consider  $S_{BKdV}(t)$  instead of  $S_{FKdV}(t)$  in Theorem 2.9. However, it is enough to prove the approximation, because  $S_{FKdV}(t)$  can be represented  $S_{BBKdV}(t)$  by Remark 2.2 and the support of initial data and  $b(k)$ . Thus, Theorem 2.9 is equivalent to the approximation between  $S_{CKdV}(t)$  and  $S_{FKdV}(t)$ . We can now reach the approximation lemma by assuming Theorem 2.8 and 2.9.

**Lemma 2.10** (Approximation lemma). *Let  $k_1, k_2 \in \mathbb{Z}^*$ ,  $A_1, A_2 > 0$ ,  $T > 0$  and  $0 < \varepsilon \ll 1$ . Then there exists a positive integer  $N_0(k_1, k_2, A_1, A_2, T, \varepsilon) \gg |k_1|, |k_2|$  such that*

$$|k_1|^{-\frac{1}{2}} |(S_{CKdV}(T)u_0)^\wedge(k_1) - (S_{FKdV}(T)u_0)^\wedge(k_1)| \leq \varepsilon,$$

and

$$|k_2|^{-\frac{1}{2}} |(S_{CKdV}(T)v_0)^\wedge(k_2) - (S_{FKdV}(T)v_0)^\wedge(k_2)| \leq \varepsilon$$

for  $N > N_0(k_1, k_2, A_1, A_2, T, \varepsilon)$  and all initial data  $u_0 \in B_{A_1}^N(0)$  and  $v_0 \in B_{A_2}^N(0)$ .

*Proof.* We assume that Theorem 2.8 and 2.9 are true for a while. The following equalities are obtained by support of the operator  $B$  and Remark 2.2,

$$(S_{FKdV}(t)u_0)^\wedge(k_1) = (BS_{FKdV}(t)u_0)^\wedge(k_1) = (S_{BBKdV}(Bu_0))^\wedge(k_1)$$

for  $|k_1| \ll N_0$ .

The constant  $\varepsilon$  in Lemma 2.10 is different from the constant  $\varepsilon$  in Theorem 2.8 and 2.9, so we let  $\varepsilon'$  denote the upper bounds in the theorems. We choose the sufficiently large



$N_0(k_1, k_2, A_1, A_2, T, \varepsilon)$  such that for all  $N > N_0$ , ' $\lesssim \varepsilon'$ ' can be changed into ' $\leq \frac{1}{2}\varepsilon$ ' in Theorem 2.8 and 2.9. Thus, we have

$$\begin{aligned} & |k_1|^{-\frac{1}{2}} |(S_{CKdV}(T) u_0)^\wedge(k_1) - (S_{BBKdV}(T) Bu_0)^\wedge(k_1)| \\ & \leq |k_1|^{-\frac{1}{2}} |(S_{CKdV}(T) u_0)^\wedge(k_1) - (S_{CKdV}(T) Bu_0)^\wedge(k_1)| \\ & + |k_1|^{-\frac{1}{2}} |(S_{CKdV}(T) Bu_0)^\wedge(k_1) - (S_{BBKdV}(T) Bu_0)^\wedge(k_1)| \\ & \leq \varepsilon, \end{aligned}$$

for  $N > N_0(k_1, k_2, A_1, A_2, T, \varepsilon)$  and  $|k_1| \leq N^{1/2}$ . In the first inequality, we use the triangle inequality. To have the second inequality, we apply Theorem 2.8 to the first term, and Theorem 2.9 to the second term, respectively. Similarly, we can obtain the estimate with respect to  $v_0$ .  $\square$

Assuming Lemma 2.10, we provide a proof of Theorem 2.6.

*Proof of Theorem 2.6.*

Let  $r_1, R_1, u_*, k_1, z$  and  $T$  as in the theorem 2.6. Choose  $0 < \varepsilon < \frac{R_1 - r_1}{2}$ , and the ball  $B_{R_1}^\infty(u_*) \subset B_{A_1}^\infty(0)$ . We also choose  $N > N_0(k_1, A_1, T, \varepsilon)$  so large that

$$\|u_* - P_{\leq N} u_*\|_{H_0^{-1/2}} \leq \varepsilon.$$

By Lemma 2.7, we can find initial data  $u_0 \in P_{\leq N} H_0^{-\frac{1}{2}}(\mathbb{T})$  satisfying

$$\|u_0 - P_{\leq N} u_*\|_{H_0^{-1/2}} \leq R_1 - \varepsilon$$

and then at time  $T$ ,

$$|k_1|^{-\frac{1}{2}} |(S_{FKdV}(T) u_0)^\wedge(k_1) - z| > r_1 + \varepsilon.$$

From the triangle inequality,

$$\|u_0 - u_*\|_{H_0^{-1/2}} \leq \|u_0 - P_{\leq N} u_*\|_{H_0^{-1/2}} + \|P_{\leq N} u_* - u_*\|_{H_0^{-1/2}} \leq R_1 - \varepsilon + \varepsilon = R_1.$$

We thus have the claim by the triangle inequality and Lemma 2.10,

$$\begin{aligned} & |k_1|^{-\frac{1}{2}} |z - (S_{CKdV}(T) u_0)^\wedge(k_1)| \\ & \geq |k_1|^{-\frac{1}{2}} [|z - (S_{FKdV}(T) u_0)^\wedge(k_1)| - |(S_{FKdV}(T) u_0)^\wedge(k_1) - (S_{CKdV}(T) u_0)^\wedge(k_1)|] \\ & > r_1 + \varepsilon - \varepsilon = r_1. \end{aligned}$$

Similarly, we also get the result for  $S_{CKdV}(t) v_0$ .  $\square$

Hence, we remain to prove Theorem 2.8 and Theorem 2.9. In Section 3, we introduce function spaces and prove bilinear and trilinear estimates. The analysis in this part is similar to [5, 7]. In Section 4, we use the normal form method via the differentiation by parts, to change the system to cubic system with bilinear boundary terms, and then we apply multilinear estimates to prove Theorem 2.8 and Theorem 2.9.

## 3. BI- AND TRILINEAR ESTIMATES

In this section, we state and prove bilinear and trilinear estimates that are used in the proof of Theorem 2.8 and 2.9. First of all, we define function spaces to obtain the multilinear estimates. These function spaces are Fourier restriction spaces that are known as the Bourgain space or the  $X^{s,b}$ -space. We slightly modify them to define  $Y^s$  and  $Z^s$  spaces for the solutions and nonlinear terms. For fixed  $s, b \in \mathbb{R}$ , and a mean-zero function  $u(x, t)$  on  $\mathbb{T} \times \mathbb{R}$ , recall

$$\|u\|_{X^{s,b}} := \left\| \langle k \rangle^s \langle \tau - k^3 \rangle^b \mathcal{F}(u)(k, \tau) \right\|_{L_k^2 L_\tau^2},$$

where  $\mathcal{F}$  is the space-time Fourier transform,

$$\mathcal{F}(u)(k, \tau) = \tilde{u}(k, \tau) = \int_{\mathbb{T} \times \mathbb{R}} e^{-i(xk + t\tau)} u(x, t) dx dt.$$

However,  $X^{s,b}$ -space barely fails to control the  $L_t^\infty H_x^s$  norm on  $\mathbb{T} \times \mathbb{R}$ . Hence, we use slightly smaller spaces by adding an additional norm,

$$\|u\|_{Y^s} := \|u\|_{X^{s,1/2}} + \|\langle k \rangle^s \mathcal{F}(u)\|_{L_k^2 L_\tau^1},$$

and the space for nonlinear terms would be

$$\|u\|_{Z^s} := \|u\|_{X^{s,-1/2}} + \left\| \frac{\langle k \rangle^s}{\langle \tau - k^3 \rangle} \mathcal{F}(u) \right\|_{L_k^2 L_\tau^1}.$$

Then, we have embeddings as follows:

$$\begin{aligned} Y^s &\subseteq C_t H_x^s \subseteq L_t^\infty H_x^s, \\ L_t^\infty H_x^s &\subseteq L_t^2 H_x^s \subseteq Z^s \end{aligned} \tag{3.1}$$

in a compact time interval  $[0, T]$ .

We introduce bilinear and trilinear terms that will appear in normal form analysis.

$$F_2(u, v) := \mathcal{F}^{-1} \left( \int \sum_{\substack{k_i \in \mathbb{Z}^* \\ k_0 + k_1 + k_2 = 0}} \frac{\tilde{u}_{k_1}}{k_1} \frac{\tilde{v}_{k_2}}{k_2} d\Gamma \right),$$

where  $\int \cdots d\Gamma$  means the integration taken on the hyperplane

$$\{(\tau_0, \tau_1, \tau_2) \in \mathbb{R}^3 : \tau_0 + \tau_1 + \tau_2 = 0\}.$$

In the analysis, we have two types of trilinear terms, namely, resonance or nonresonance terms:

$$F_r(u, v, w) := \mathcal{F}^{-1} \left( \int \tilde{u}_{-k} \sum_{\substack{k_i \in \mathbb{Z}^* \\ k_2 + k_3 = 0}} \tilde{v}_{k_2} \frac{\tilde{w}_{k_3}}{k_3} d\Gamma \right), \tag{3.2}$$

and

$$F_{nr}(u, v, w) := \mathcal{F}^{-1} \left( \int \sum_{\substack{k_i \in \mathbb{Z}^* \\ k_0 + k_1 + k_2 + k_3 = 0 \\ (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0}} \tilde{u}_{k_1} \tilde{v}_{k_2} \frac{\tilde{w}_{k_3}}{k_3} d\Gamma \right),$$

the integral  $\int \cdots d\Gamma$  is taken on the set

$$\{(\tau_0, \tau_1, \tau_2, \tau_3) \in \mathbb{R}^4 : \tau_0 + \tau_1 + \tau_2 + \tau_3 = 0\}.$$

For dyadic numbers  $N_i$ , we assume  $N_i \sim |k_i|$ . We denote by  $n_1, n_2, n_3, n_4 \in \mathbb{Z}$  frequencies in order, i.e.,

$$|n_1| \geq |n_2| \geq |n_3| \geq |n_4| \quad \text{and} \quad \{n_1, n_2, n_3, n_4\} = \{k_0, k_1, k_2, k_3\}.$$

Similarly, in the case of three frequencies, let  $n_1, n_2$  and  $n_3$  be defined to be the maximum, median and minimum of  $k_0, k_1$  and  $k_2$ , respectively. Namely,

$$n_1 \geq n_2 \geq n_3 \quad \text{and} \quad \{n_1, n_2, n_3\} = \{k_0, k_1, k_2\}.$$

### 3.1. Bilinear estimate.

**Lemma 3.1.** *Let  $u, v \in Y^{-1/2}$ . Then*

$$\|\partial_x^{-1} u \partial_x^{-1} v\|_{Y^{-1/2}} \lesssim \|u\|_{Y^{-1/2}} \|v\|_{Y^{-1/2}}. \quad (3.3)$$

*Proof.* The  $X^{-1/2, 1/2}$  part of (3.3) is a variant of

$$\|uv\|_{X^{-1/2, 1/2}} \lesssim \|u\|_{Y^{1/2}} \|v\|_{Y^{1/2}}, \quad (3.4)$$

which was proved in Section 4 of [5], so it is done. To prove the  $L_k^2 L_\tau^1$  part, it is enough to show the estimate

$$\|uv\|_{L_x^2 L_\tau^1} \lesssim \|u\|_{H_x^{1/2} L_\tau^1} \|v\|_{H_x^{1/2} L_\tau^1}.$$

It can be obtained by the Young, Hölder, and Sobolev inequalities.  $\square$

### 3.2. Trilinear estimate.

In this subsection, we prove the trilinear estimate of the following form.

**Lemma 3.2.** *Let  $u, v$  and  $w \in Y^{-1/2}$ . Then*

$$\|(uv - P_0(uv)) \partial_x^{-1} w\|_{Z^{-1/2}} \lesssim \|u\|_{Y^{-1/2}} \|v\|_{Y^{-1/2}} \|w\|_{Y^{-1/2}}, \quad (3.5)$$

where  $P_0$  is the Dirichlet projection to zero frequency, i.e.,  $P_0(f) := \int_{\mathbb{T}} f dx$ .

As opposed to the bilinear estimate, the trilinear term contains resonant interactions. We decompose it into resonant part and nonresonant part. We first consider the resonant part.

**Lemma 3.3.** *Let  $u, v$  and  $w \in Y^{-1/2}$ . We have*

$$\|F_r(u, v, w)\|_{Z^{-1/2}} \lesssim \|u\|_{Y^{-1/2}} \|v\|_{Y^{-1/2}} \|w\|_{Y^{-1/2}}.$$

*Proof.* To prove the lemma, we handle the space variable and the time variable in consecutive order. We first show an estimate for the spatial domain,

$$\|F_r(u, v, w)\|_{H_x^{-1/2}} \lesssim \|u\|_{H_x^{-1/2}} \|v\|_{H_x^{-1/2}} \|w\|_{H_x^{-1/2}}.$$

By duality and the Plancherel's, it suffices to prove the estimate

$$\left| \int z u dx \right| \left| \int v W dx \right| \lesssim \|z\|_{H_x^{1/2}} \|u\|_{H_x^{-1/2}} \|v\|_{H_x^{-1/2}} \|W\|_{H_x^{1/2}}, \quad (3.6)$$

where  $W = \partial_x^{-1} w$ . It is deduced by the Hölder inequality.

For the time variable, it is obvious that

$$\left\| \langle \tau - k^3 \rangle^{-1/2} \mathcal{F}[F_r(u, v, w)] \right\|_{L_{k, \tau}^2} \lesssim \|F_r(u, v, w)\|_{L_{x, t}^2},$$

and

$$\left\| \langle \tau - k^3 \rangle^{-1} \mathcal{F}(F_r(u, v, w)) \right\|_{L_k^2 L_\tau^1} \lesssim \|F_r(u, v, w)\|_{L_{x, t}^2}.$$

By taking temporal frequency translation  $e^{-t\partial_x^3}$ , the claim is reduced to

$$\|F_r(u, v, w)\|_{L_t^2 H_x^{-1/2}} \lesssim \|u\|_{H_t^{1/2} H_x^{-1/2}} \|v\|_{H_t^{1/2} H_x^{-1/2}} \|w\|_{H_t^{1/2} H_x^{-1/2}},$$

and then the claim follows from (3.6), the Hölder inequality, and the Sobolev inequality.  $\square$

Next, we consider the nonresonant case. We can prove a slightly stronger estimate for the nonresonant part.

**Lemma 3.4.** *Let  $u, v$  and  $w \in Y^{-1/2}$  and  $N_0, N_1, N_2$  and  $N_3$  be dyadic numbers. Then*

$$\|P_{N_0} F_{nr}(P_{N_1} u, P_{N_2} v, P_{N_3} w)\|_{Z^{-1/2}} \lesssim \left( \frac{N_i}{n_1} \right)^\sigma n_3^{-\sigma} \|u\|_{Y^{-1/2}} \|v\|_{Y^{-1/2}} \|w\|_{Y^{-1/2}}, \quad (3.7)$$

for small enough  $\sigma > 0$  and  $i = 1$  or  $2$ .

A part of proof of the estimate (3.7) relies on

$$\|uvw\|_{L_{x, t}^2} \lesssim \|u\|_{X^{0, 1/2-\delta}} \|v\|_{X^{0, 1/2-\delta}} \|w\|_{X^{1/2-\delta, 1/2-\delta}}, \quad (3.8)$$

for some small  $0 < \delta \ll 1$ . For the proof, see Section 7 in [5].

*Proof.* By symmetry, we assume that  $i = 1$ . We first prove the  $X^{-1/2, -1/2}$  part, i.e.,

$$\|P_{N_0} F_{nr}(P_{N_1} u, P_{N_2} v, P_{N_3} w)\|_{X^{-1/2, -1/2}} \lesssim \left( \frac{N_1}{n_1} \right)^\sigma n_3^{-\sigma} \|u\|_{X^{-1/2, 1/2}} \|v\|_{X^{-1/2, 1/2}} \|w\|_{X^{-1/2, 1/2}}.$$

Without loss of generality, we may assume that all  $u_i$  are nonnegative. By duality, it is equivalent to

$$\begin{aligned} & \left| \iint u_0 F_{nr}(u_1 u_2 u_3) dx dt \right| \\ & \lesssim \left( \frac{N_1}{n_1} \right)^\sigma n_3^{-\sigma} \|u_0\|_{X^{1/2, 1/2}} \|u_1\|_{X^{-1/2, 1/2}} \|u_2\|_{X^{-1/2, 1/2}} \|u_3\|_{X^{1/2, 1/2}}, \end{aligned} \quad (3.9)$$

where  $U_3 = \partial_x^{-1} u_3$  and  $u_i$  has Fourier support on the region  $|k_i| \sim N_i$ . The right hand side of (3.9) is comparable to

$$\left(\frac{N_1}{n_1}\right)^\sigma n_3^{-\sigma} \frac{|N_0 N_3|^{1/2}}{|N_1 N_2|^{1/2}} \|u_0\|_{X^{0,1/2}} \|u_1\|_{X^{0,1/2}} \|u_2\|_{X^{0,1/2}} \|U_3\|_{X^{0,1/2}}. \quad (3.10)$$

**Lemma 3.5.** *In the same notation, we have*

$$\left(\frac{N_1}{n_1}\right)^\sigma \frac{|N_0 N_3|^{1/2}}{|N_1 N_2|^{1/2}} \geq \frac{n_3^{1/2} n_4^{1/2}}{n_1}. \quad (3.11)$$

*Proof.* If  $N_1 \sim n_1$ , then we can easily obtain (3.11). Hence we may assume that  $N_1 \ll n_1$ . We rewrite (3.11) as

$$\frac{n_3^{1/2} n_4^{1/2}}{n_1} \left(\frac{n_1}{N_1}\right)^\sigma \frac{|N_1 N_2|^{1/2}}{|N_0 N_3|^{1/2}} \leq 1.$$

In other words,

$$\frac{|N_1|^{1/2-\sigma}}{n_1^{1/2-\sigma}} \frac{|N_2|^{1/2}}{n_1^{1/2}} \frac{n_3^{1/2} n_4^{1/2}}{|N_0 N_3|^{1/2}} \leq 1.$$

Each term of the left hand side is smaller than 1, and so we are done.  $\square$

From Lemma 3.5, (3.10) is bounded below by

$$\frac{n_3^{1/2-\sigma} n_4^{1/2}}{n_1} \|u_0\|_{X^{0,1/2}} \|u_1\|_{X^{0,1/2}} \|u_2\|_{X^{0,1/2}} \|U_3\|_{X^{0,1/2}}. \quad (3.12)$$

By a resonance identity,

$$\sum_{i=0,1,2,3} (\tau_i - k_i^3) = - \sum_{i=0,1,2,3} k_i^3 = 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1),$$

and thus,

$$\sup_{i=0,1,2,3} L_i \gtrsim |k_1 + k_2| |k_2 + k_3| |k_3 + k_1|,$$

for  $L_i = \langle \tau_i - k_i^3 \rangle$ . Due to the symmetry of the functions in (3.12), we only consider that  $L_0 = \sup_{i=0,1,2,3} L_i$ , and by Lemma 4.4 of [6], we have

$$L_0 \gtrsim n_1^2 n_4^{-1}.$$

Therefore, to prove (3.9), it suffices to show that

$$\left| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int n_3^{-1/2+\sigma} L_0^{1/2} \tilde{u}_0 \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right| \lesssim \|u_0\|_{X^{0,1/2}} \|u_1\|_{X^{0,1/2}} \|u_2\|_{X^{0,1/2}} \|U_3\|_{X^{0,1/2}}.$$

Here, as  $\int \cdots d\Gamma$  is the integral taken over the set  $\{\tau_0 + \tau_1 + \tau_2 + \tau_3 = 0\}$  before, and we denote  $|\mathbf{k}| = (k_0, k_1, k_2, k_3)$  and  $\mathcal{NR} = \{(k_0, k_1, k_2, k_3) : k_0 + k_1 + k_2 + k_3 = 0, (k_1 + k_2)(k_2 + k_3)(k_3 + k_1) \neq 0\}$ .

At least one of  $k_1$ ,  $k_2$  and  $k_3$  is  $\mathcal{O}(n_3)$ . By symmetry, let us suppose that it is  $k_3$ . Then it is enough to show that

$$\left| \iint u_0 u_1 u_2 U_3 dx dt \right| \lesssim \|u_0\|_{L_{x,t}^2} \|u_1\|_{X^{0,1/2}} \|u_2\|_{X^{0,1/2}} \|U_3\|_{X^{1/2-\sigma,1/2}}. \quad (3.13)$$

From the Cauchy-Schwarz inequality and (3.8), we can get (3.13) for sufficiently small  $\sigma$ .

Next, we prove the  $L_k^2 L_\tau^1$  part. From the Hölder inequality,  $X^{-1/2,-1/2}$  part and the interpolation, it is enough to prove the estimate

$$\left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{1}{|k_0|^{1/2} L_0^{1-\delta}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_0}^1} \lesssim \|u_1\|_{X^{-1/2,1/2}} \|u_2\|_{X^{-1/2,1/2}} \|U_3\|_{X^{1/2,1/2}},$$

where  $U_3 = \partial_x^{-1} u_3$  and some positive constant  $\delta \ll 1$ . In other words, we will show that

$$\left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{|k_1 k_2|^{1/2}}{|k_0 k_3|^{1/2} L_0^{1-\delta} L_1^{1/2} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_0}^1} \lesssim \|u_1\|_{L_{x,t}^2} \|u_2\|_{L_{x,t}^2} \|U_3\|_{L_{x,t}^2}. \quad (3.14)$$

From the Hölder inequality, the left hand side of (3.14) is bounded by

$$\left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{|k_1 k_2|^{1/2}}{|k_0 k_3|^{1/2} L_0^{1/2-2\delta} L_1^{1/2} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_0}^2}. \quad (3.15)$$

Similarly to  $X^{-1/2,-1/2}$  case, we have  $\sup_{i=0,1,2,3} L_i (= L_s) \gtrsim n_1^2 n_4^{-1}$ , and so

$$\frac{|k_1 k_2|^{1/2}}{|k_0 k_3|^{1/2} L_s^{1/2}} \lesssim \frac{n_1}{k_0^{1/2} k_3^{1/2}} \cdot \frac{n_4^{1/2}}{n_1} \leq \frac{1}{k_3^{1/2}}. \quad (3.16)$$

We first consider  $\sup_{i=1,2,3} L_i = L_1$ . By combining (3.15) and (3.16), it suffices to show that

$$\left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{1}{k_3^{1/2} L_0^{1/2-2\delta} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_0}^2} \lesssim \|u_1\|_{L_{x,t}^2} \|u_2\|_{L_{x,t}^2} \|U_3\|_{L_{x,t}^2}. \quad (3.17)$$

Then by duality, it suffices to prove

$$\left| \iint u_0 u_1 u_2 U_3 dx dt \right| \lesssim \|u_0\|_{X^{0,1/2-2\delta}} \|u_1\|_{L_{x,t}^2} \|u_2\|_{X^{0,1/2}} \|U_3\|_{X^{1/2,1/2}}. \quad (3.18)$$

We can obtain (3.18) by the Cauchy-Schwarz inequality and (3.8) for small enough  $\delta > 0$ . By symmetry,  $\sup_{i=0,1,2,3} L_i = L_2$  or  $L_3$  cases are proved as well.

Finally, we assume that  $\sup_{i=0,1,2,3} L_i = L_0 \gtrsim n_1^2 n_4^{-1}$ . From (3.16), the left hand side of

(3.14) is bounded by

$$\left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{1}{n_3^{1/2} L_0^{1/2-\delta} L_1^{1/2} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_0}^1}. \quad (3.19)$$

From the assumption  $L_0 \gtrsim L_1$ , the Fubini theorem and the Cauchy-Schwarz inequality,

$$\begin{aligned} (3.19) &\leq \left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{1}{n_3^{1/2} L_1^{1-\delta} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_1}^1} \\ &\lesssim \left\| \sum_{|\mathbf{k}| \in \mathcal{NR}} \int \frac{1}{n_3^{1/2} L_1^{1/2-2\delta} L_2^{1/2} L_3^{1/2}} \tilde{u}_1 \tilde{u}_2 \tilde{U}_3 d\Gamma \right\|_{L_{k_0}^2 L_{\tau_1}^2}. \end{aligned} \quad (3.20)$$

The last term of (3.20) is similar to the left hand side of (3.17), so we finish the proof.  $\square$

*Proof of Lemma 3.2.* It is obtained from the combination of Lemma 3.3 and 3.4 with summation with respect to each of dyadic frequency supports. More precisely, we observe

$$(uv - P_0(uv)) \partial_x^{-1} w = F_r(u, v, w) + F_r(v, u, w) + F_{nr}(u, v, w),$$

and use the fact that left hand side of (3.7) vanishes unless  $n_1 \sim n_2$  when we sum up with dyadic numbers.  $\square$

**Remark 3.6.** *In the proof of main theorems, we will use a rescaling argument. The bilinear and trilinear estimates obtained above can be easily restated with a rescaling parameter. We record facts here for convenience of readers. See more details in [2], [4], [5] and [17]. We let  $\alpha\mathbb{T} = [0, 2\pi\alpha)$  be the spatial domain. Then implicit constants of (3.4) and (3.8) depend on  $\alpha$ . More precisely, for  $2\pi\alpha$ -periodic function  $u$ , we define*

$$\|u\|_{H^s(\alpha\mathbb{T})} := \frac{1}{(2\pi\alpha)^{1/2}} \left( \sum_{k \in \mathbb{Z}/\alpha} \langle k \rangle^{2s} |\hat{u}|^2 \right)^{1/2}, \quad \hat{u}(k) = \int_0^{2\pi\alpha} e^{-ikx} u(x) dx \quad (3.21)$$

and

$$\|u\|_{X^{s,b}(\alpha\mathbb{T})} := \left\| \langle k \rangle^s \langle \tau - k^3 \rangle^b \mathcal{F}(u)(k, \tau) \right\|_{L_k^2(\mathbb{Z}/\alpha) L_\tau^2}.$$

In addition, we can define  $Y^s(\alpha\mathbb{T})$  and  $Z^s(\alpha\mathbb{T})$  norm by the same method. From [5], we have

$$\|uv\|_{X^{-1/2,1/2}(\alpha\mathbb{T})} \lesssim \alpha^{0+} \|u\|_{Y^{1/2}(\alpha\mathbb{T})} \|v\|_{Y^{1/2}(\alpha\mathbb{T})}. \quad (3.22)$$

Moreover, the following estimates are well-known,

$$\|u\|_{L_{x,t}^4(\alpha\mathbb{T})} \lesssim C(\alpha) \|u\|_{X^{0,1/3}(\alpha\mathbb{T})} \quad (3.23)$$

and

$$\|u\|_{L_{x,t}^\infty(\alpha\mathbb{T})} \lesssim C(\alpha) \|u\|_{X^{\frac{1}{2}+, \frac{1}{2}+}(\alpha\mathbb{T})}, \quad (3.24)$$

where implicit constants  $C(\alpha)$  are decreasing functions of  $\alpha$ . In particular, we have  $C(\alpha) \leq C(1)$  for  $\alpha \geq 1$ . From (3.23) and (3.24),

$$\|uvw\|_{L_{x,t}^2(\alpha\mathbb{T})} \lesssim C'(\alpha) \|u\|_{X^{0,1/3}(\alpha\mathbb{T})} \|v\|_{X^{0,1/3}(\alpha\mathbb{T})} \|w\|_{X^{\frac{1}{2}+, \frac{1}{2}+}(\alpha\mathbb{T})}, \quad (3.25)$$

where implicit constant  $C'(\alpha)$  is also a decreasing function of  $\alpha$ . Moreover, by rescaling (3.8), we can obtain

$$\|uvw\|_{L^2_{x,t}(\alpha\mathbb{T})} \lesssim \alpha^M \|u\|_{X^{0,1/2-\delta}(\alpha\mathbb{T})} \|v\|_{X^{0,1/2-\delta}(\alpha\mathbb{T})} \|w\|_{X^{1/2-\delta,1/2-\delta}(\alpha\mathbb{T})}, \quad (3.26)$$

for some positive constant  $M$ . Interpolating (3.25) and (3.26) we can obtain the  $\alpha$ -rescaled estimate as follows

$$\|uvw\|_{L^2_{x,t}(\alpha\mathbb{T})} \lesssim \alpha^{0+} \|u\|_{X^{0,1/2-\sigma}(\alpha\mathbb{T})} \|v\|_{X^{0,1/2-\sigma}(\alpha\mathbb{T})} \|w\|_{X^{1/2-\sigma,1/2-\sigma}}, \quad (3.27)$$

for some small  $0 < \sigma \ll 1$ . Once we obtain (3.21)-(3.23) and (3.27), it is straightforward that one can replace (3.4) and (3.8) in the proof, and so conclude (3.3) and (3.5) with the scaling parameter  $\alpha$ .

#### 4. DIFFERENTIATION BY PARTS AND PROOF OF THEOREM 2.8 AND 2.9

In this section, we use the method of the normal form to show Theorem 2.8 and 2.9. The normal form is performed via the differentiation by parts. Writing the system (CKdV) in the interaction representation, we take the differentiation by parts to change the quadratic nonlinear terms into the bilinear nonlinear terms as the boundary term and the trilinear terms. This procedure replaces the use of the Miura transform in the proof of [6].

**4.1. Differentiation by parts.** To simplify the notation, we denote

$$\begin{aligned} u(t, x) &= S_{CKdV}(t) u_0, & v(t, x) &= S_{CKdV}(t) v_0, \\ u^b(t, x) &= S_{BKdV}(t) u_0, & v^b(t, x) &= S_{BKdV}(t) v_0. \end{aligned}$$

Moreover, denote  $\mathbf{u} = e^{t\partial_x^3} u$ ,  $\mathbf{v} = e^{t\partial_x^3} v$ ,  $\mathbf{u}^b = e^{t\partial_x^3} u^b$  and  $\mathbf{v}^b = e^{t\partial_x^3} v^b$ . From (CKdV),

$$\begin{aligned} \partial_t \mathbf{u} &= e^{t\partial_x^3} (\partial_x^3 u + \partial_t u) = -\frac{e^{t\partial_x^3}}{2} \partial_x (vv) = -\frac{e^{t\partial_x^3}}{2} \partial_x \left( e^{-t\partial_x^3} \mathbf{v} \cdot e^{-t\partial_x^3} \mathbf{v} \right), \\ \partial_t \mathbf{v} &= e^{t\partial_x^3} (\partial_x^3 v + \partial_t v) = -e^{t\partial_x^3} \partial_x (uv) = -e^{t\partial_x^3} \partial_x \left( e^{-t\partial_x^3} \mathbf{u} \cdot e^{-t\partial_x^3} \mathbf{v} \right). \end{aligned}$$

We look at the system of the Fourier variables, still denoted as  $\mathbf{u}_k$ ,  $\mathbf{v}_k$  for  $k \in \mathbb{Z}^*$ ,<sup>2</sup>

$$\begin{aligned} \partial_t \mathbf{u}_k &= -\frac{i}{2} e^{-ik^3 t} \sum_{k_1+k_2=k} k e^{ik_1^3 t} \mathbf{v}_{k_1} e^{ik_2^3 t} \mathbf{v}_{k_2} = -\frac{i}{2} \sum_{k_1+k_2=k} k e^{-i\phi(k)t} \mathbf{v}_{k_1} \mathbf{v}_{k_2}, \\ \partial_t \mathbf{v}_k &= -ie^{-ik^3 t} \sum_{k_1+k_2=k} k e^{ik_1^3 t} \mathbf{u}_{k_1} e^{ik_2^3 t} \mathbf{v}_{k_2} = -i \sum_{k_1+k_2=k} k e^{-i\phi(k)t} \mathbf{u}_{k_1} \mathbf{v}_{k_2}, \end{aligned}$$

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<sup>2</sup> $k_i \neq 0$  in the sequel is due to mean zero assumption.



where  $\phi(k) = \phi(k_1, k_2) = 3k_1k_2(k_1 + k_2)$ . Taking the differentiation by parts, we write

$$\begin{aligned}
 \partial_t \mathbf{u}_k &= -\frac{i}{2} \left[ \partial_t \left\{ \sum_{k_1+k_2=k} k \frac{e^{-i\phi(k)t}}{-i\phi(k)} \mathbf{v}_{k_1} \mathbf{v}_{k_2} \right\} \right. \\
 &\quad \left. + 2 \sum_{k_1+k_3=k} k \frac{e^{-i\phi(k_1, k_3)t}}{-i\phi(k_1, k_3)} \mathbf{v}_{k_3} \sum_{k_{11}+k_{12}=k_1} ik_1 e^{-i\phi(k_{11}, k_{12})t} \mathbf{u}_{k_{11}} \mathbf{v}_{k_{12}} \right] \\
 &= \frac{1}{6} \left[ \partial_t \left\{ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1k_2} \mathbf{v}_{k_1} \mathbf{v}_{k_2} \right\} + 2i \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{e^{-i\Phi(k)t}}{k_3} \mathbf{u}_{k_1} \mathbf{v}_{k_2} \mathbf{v}_{k_3} \right] \\
 \partial_t \mathbf{v}_k &= -i \left[ \partial_t \left\{ \sum_{k_1+k_2=k} k \frac{e^{-i\phi(k)t}}{-i\phi(k)} \mathbf{u}_{k_1} \mathbf{v}_{k_2} \right\} \right. \\
 &\quad + \sum_{k_1+k_3=k} k \frac{e^{-i\phi(k_1, k_3)t}}{-i\phi(k_1, k_3)} \mathbf{u}_{k_1} \sum_{k_{21}+k_{22}=k_3} ik_3 e^{-i\phi(k_{21}, k_{22})t} \mathbf{u}_{k_{21}} \mathbf{v}_{k_{22}} \\
 &\quad \left. + \frac{1}{2} \sum_{k_1+k_3=k} k \frac{e^{-i\phi(k_1, k_3)t}}{-i\phi(k_1, k_3)} \mathbf{v}_{k_3} \sum_{k_{11}+k_{12}=k_1} ik_1 e^{-i\phi(k_{11}, k_{12})t} \mathbf{v}_{k_{11}} \mathbf{v}_{k_{12}} \right] \\
 &= \frac{1}{3} \left[ \partial_t \left\{ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1k_2} \mathbf{u}_{k_1} \mathbf{v}_{k_2} \right\} + \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{ie^{-i\Phi(k)t}}{k_3} \left( \mathbf{v}_{k_1} \mathbf{u}_{k_2} \mathbf{u}_{k_3} + \frac{1}{2} \mathbf{v}_{k_1} \mathbf{v}_{k_2} \mathbf{v}_{k_3} \right) \right], \tag{4.1}
 \end{aligned}$$

where  $\Phi(k) = \Phi(k_1, k_2, k_3) = 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)$ . Similarly, we write a system for (BKdV),

$$\begin{aligned}
 \partial_t \mathbf{u}_k^b &= \frac{1}{6} \left[ \partial_t \left\{ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1k_2} b(k) \mathbf{v}_{k_1}^b \mathbf{v}_{k_2}^b \right\} + 2i \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{e^{-i\Phi(k)t}}{k_3} b(k) b(k_1 + k_2) \mathbf{u}_{k_1}^b \mathbf{v}_{k_2}^b \mathbf{v}_{k_3}^b \right] \\
 \partial_t \mathbf{v}_k^b &= \frac{1}{3} \left[ \partial_t \left\{ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1k_2} b(k) \mathbf{v}_{k_1}^b \mathbf{v}_{k_2}^b \right\} \right. \\
 &\quad \left. + \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{ie^{-i\Phi(k)t}}{k_3} b(k) b(k_1 + k_2) \left( \mathbf{v}_{k_1}^b \mathbf{u}_{k_2}^b \mathbf{u}_{k_3}^b + \frac{1}{2} \mathbf{v}_{k_1}^b \mathbf{v}_{k_2}^b \mathbf{v}_{k_3}^b \right) \right]. \tag{4.2}
 \end{aligned}$$

Integrating (4.1) and (4.2) in time  $t$ , we have

$$\begin{aligned} \mathbf{u}_k(t) = & \mathbf{u}_k(0) + \frac{1}{6} \left[ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1 k_2} \mathbf{v}_{k_1}(t) \mathbf{v}_{k_2}(t) - \sum_{k_1+k_2=k} \frac{1}{k_1 k_2} \mathbf{v}_{k_1}(0) \mathbf{v}_{k_2}(0) \right. \\ & \left. + 2i \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{e^{-i\Phi(k)s}}{k_3} \mathbf{u}_{k_1} \mathbf{v}_{k_2} \mathbf{v}_{k_3} ds \right] \end{aligned} \quad (4.3)$$

$$\begin{aligned} \mathbf{v}_k(t) = & \mathbf{v}_k(0) + \frac{1}{3} \left[ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1 k_2} \mathbf{u}_{k_1}(t) \mathbf{v}_{k_2}(t) - \sum_{k_1+k_2=k} \frac{1}{k_1 k_2} \mathbf{u}_{k_1}(0) \mathbf{v}_{k_2}(0) \right. \\ & \left. + \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{ie^{-i\Phi(k)s}}{k_3} \left( \mathbf{v}_{k_1} \mathbf{u}_{k_2} \mathbf{u}_{k_3} + \frac{1}{2} \mathbf{v}_{k_1} \mathbf{v}_{k_2} \mathbf{v}_{k_3} \right) ds \right] \end{aligned} \quad (4.4)$$

$$\begin{aligned} \mathbf{u}_k^b(t) = & \mathbf{u}_k(0) + \frac{1}{6} \left[ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1 k_2} b(k) \mathbf{v}_{k_1}^b(t) \mathbf{v}_{k_2}^b(t) - \sum_{k_1+k_2=k} \frac{1}{k_1 k_2} b(k) \mathbf{v}_{k_1}(0) \mathbf{v}_{k_2}(0) \right. \\ & \left. + 2i \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{e^{-i\Phi(k)s}}{k_3} b(k) b(k_1+k_2) \mathbf{u}_{k_1}^b \mathbf{v}_{k_2}^b \mathbf{v}_{k_3}^b ds \right] \end{aligned} \quad (4.5)$$

$$\begin{aligned} \mathbf{v}_k^b(t) = & \mathbf{v}_k(0) + \frac{1}{3} \left[ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1 k_2} b(k) \mathbf{u}_{k_1}^b(t) \mathbf{v}_{k_2}^b(t) - \sum_{k_1+k_2=k} \frac{1}{k_1 k_2} b(k) \mathbf{u}_{k_1}(0) \mathbf{v}_{k_2}(0) \right. \\ & \left. + \int_0^s \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{ie^{-i\Phi(k)s}}{k_3} b(k) b(k_1+k_2) \left( \mathbf{v}_{k_1}^b \mathbf{u}_{k_2}^b \mathbf{u}_{k_3}^b + \frac{1}{2} \mathbf{v}_{k_1}^b \mathbf{v}_{k_2}^b \mathbf{v}_{k_3}^b \right) ds \right]. \end{aligned} \quad (4.6)$$

Transforming back (4.3), we write  $u_k(t)$  as follows,

$$\begin{aligned} u_k(t) = & e^{ik^3 t} u_k(0) + \frac{e^{ik^3 t}}{6} \left[ \sum_{k_1+k_2=k} \frac{e^{-i\phi(k)t}}{k_1 k_2} e^{-it(k_1^3+k_2^3)} v_{k_1}(t) v_{k_2}(t) - \sum_{k_1+k_2=k} \frac{v_{k_1}(0) v_{k_2}(0)}{k_1 k_2} \right. \\ & \left. + 2i \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{e^{-i\Phi(k)s}}{k_3} e^{-is(k_1^3+k_2^3+k_3^3)} u_{k_1} v_{k_2} v_{k_3} ds \right]. \end{aligned} \quad (4.7)$$

In the same way, the solutions  $v_k(t)$ ,  $u_k^b(t)$  and  $v_k^b(t)$  are denoted similarly. As shown in the theorems, we should investigate in detail the difference between solutions. Since  $v$  and  $v^b$  are handled similarly, we mainly consider the solutions  $u$  and  $u^b$ . Note that ‘ $k_1 + k_2 \neq 0$ ’ in (4.7) means ‘ $-P_0(uv) \partial_x^{-1} v$ ’ in the spatial domain. This represent why we require the trilinear form in Lemma 3.2 in the approximation analysis.

#### 4.2. Proof of Theorem 2.9.

We prove the estimate (2.3) for  $u - u_b$ . By the following argument, we can obtain (2.3) for  $v - v_b$  as well. From (3.1) and (4.3)-(4.7), the first term of the left hand side of (2.3) is bounded by

$$\begin{aligned} & \left\| P_{\leq N^{1/2}} (u - u^b) \right\|_{Y^{-1/2}} \\ & \lesssim \left\| P_{\leq N^{1/2}} \left[ \partial_x^{-1} v \partial_x^{-1} v - \partial_x^{-1} v^b \partial_x^{-1} v^b \right] \right\|_{Y^{-1/2}} \\ & + \left\| P_{\leq N^{1/2}} \left[ (uv - P_0(uv)) \partial_x^{-1} v - \left( B(u^b v^b) - P_0(B(u^b v^b)) \right) \partial_x^{-1} v^b \right] \right\|_{Z^{-1/2}} \\ & =: \|B_2(v, v)\|_{Y^{-1/2}} + \|N_3(u, v, v)\|_{Z^{-1/2}}. \end{aligned} \quad (4.8)$$

We now use the bilinear and trilinear estimates obtained in Section 3. From the triangle inequality and (3.3),

$$\begin{aligned} \|B_2(v, v)\|_{Y^{-1/2}} & \leq \left\| P_{\leq N^{1/2}} \left[ \partial_x^{-1} v \partial_x^{-1} (v - v^b) \right] \right\|_{Y^{-1/2}} + \left\| P_{\leq N^{1/2}} \left[ \partial_x^{-1} v^b \partial_x^{-1} (v - v^b) \right] \right\|_{Y^{-1/2}} \\ & \lesssim \|P_{\leq N^{1/2}} v\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} (v - v^b)\|_{Y^{-1/2}} \\ & + \|P_{\leq N^{1/2}} v^b\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} (v - v^b)\|_{Y^{-1/2}} \\ & + (\text{remainder terms})_1. \end{aligned} \quad (4.9)$$

Here  $(\text{remainder terms})_1$  contain *high-high* to *low* frequency interactions.

The integral terms are also estimated by the triangle inequality, (3.5) and  $P_0 B = P_0$ ,

$$\begin{aligned} \|N_3(u, v, v)\|_{Z^{-1/2}} & \lesssim \|P_{\leq N^{1/2}} u\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} v\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} (v - v^b)\|_{Y^{-1/2}} \\ & + \|P_{\leq N^{1/2}} u\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} v^b\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} (v - v^b)\|_{Y^{-1/2}} \\ & + \|P_{\leq N^{1/2}} v^b\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} v^b\|_{Y^{-1/2}} \|P_{\leq N^{1/2}} (u - u^b)\|_{Y^{-1/2}} \\ & + \left\| P_{\leq N^{1/2}} \mathcal{F}_x^{-1} \left[ \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{(1-b(k_1+k_2))}{k_3} u_{k_1}^b v_{k_2}^b v_{k_3}^b \right] \right\|_{Z^{-1/2}} \\ & + (\text{remainder terms})_2. \end{aligned} \quad (4.10)$$

We take three steps. We first show that the remainder terms are  $\mathcal{O}_N(1)$  using the bilinear and trilinear estimates that is obtained the last section. Next, we show that  $Z^{-1/2}$ -term of (4.10) is  $\mathcal{O}_N(1)$  as well. Lastly, we show that terms involving the difference are absorbed into the left hand side of (4.8) and  $\|P_{\leq N^{1/2}} (v - v^b)\|_{Y^{-1/2}}$ . For this step, we use a rescaling

argument to make the factor  $\|P_{\leq N^{1/2}}u\|_{Y^{-1/2}}$  small in a large domain.

**Step 1.**

First, we handle (remainder terms)<sub>1</sub> in the boundary terms. We take a dyadic decomposition and use Lemma 3.1, Lemma 3.3 and Lemma 3.4. In view of (4.8), (remainder terms)<sub>1</sub> contains only *high-high* to *low* interactions. Namely, it is bounded by

$$\|P_{\leq N^{1/2}}F_2(v_{hi}, v_{hi})\|_{Y^{-1/2}},$$

where we denote  $v_{low} = P_{\leq N^{1/2}}v$  and  $v_{hi} = (1 - P_{\leq N^{1/2}})v$ . Obviously,  $v$  can be replaced to  $u$ ,  $u^b$  or  $v^b$ , but they are handled in the same way. From Lemma 3.1 and the global well-posedness, since

$$\|F_2(v, v)\|_{Y^{-1/2}} \lesssim \|v\|_{Y^{-1/2}}\|v\|_{Y^{-1/2}},$$

we have

$$\|P_{\leq N^{1/2}}F_2(v_{hi}, v_{hi})\|_{Y^{-1/2}} \lesssim \|v_{hi}\|_{Y^{-1/2}}\|v_{hi}\|_{Y^{-1/2}} \sim \mathcal{O}_N(1). \quad (4.11)$$

Next, we control the integral terms. Similarly, (remainder terms)<sub>2</sub> also has the sum of the multilinear terms, but it has resonant form  $P_{\leq N^{1/2}}F_r(u, v, v)$  and nonresonant form  $P_{\leq N^{1/2}}F_{nr}(u, v, v)$ . The resonant case can be controlled as the boundary terms did. More precisely, we write the  $P_{\leq N^{1/2}}F_r(u, v, v)$  as follows,

$$\sum_{N_0, N_1, N_2, N_3} P_{N_0} P_{\leq N^{1/2}} F_r(P_{N_1}u, P_{N_2}v, P_{N_3}v). \quad (4.12)$$

Likewise,  $u$  and  $v$  in the sequel can be replaced by  $u^b$  or  $v^b$ . As before, in view of (4.10), (remainder terms)<sub>2</sub> does not contain trilinear terms of which all factors are from low frequency piece. We thus have  $n_1 > N^{1/2}$  and from (3.2), we can write the form (4.12) in (remainder terms)<sub>2</sub> as follows,

$$\sum_{N_0, N_2} P_{N_0} P_{\leq N^{1/2}} F_r(P_{N_0}u_{low}, P_{N_2}v_{hi}, P_{N_2}v_{hi}) = P_{\leq N^{1/2}} F_r(u_{low}, v_{hi}, v_{hi}). \quad (4.13)$$

Similarly to the boundary case, we have the following estimates by Lemma 3.3 and the global well-posedness,

$$\|F_r(u_{low}, v, v)\|_{Z^{-1/2}} \lesssim 1,$$

and

$$\|P_{\leq N^{1/2}}F_r(u_{low}, v_{hi}, v_{hi})\|_{Z^{-1/2}} \lesssim \|u\|_{Y^{-1/2}}\|v_{hi}\|_{Y^{-1/2}}\|v_{hi}\|_{Y^{-1/2}} \sim \mathcal{O}_N(1). \quad (4.14)$$

In other words, (remainder terms)<sub>2</sub> is bounded by  $\mathcal{O}_N(1)$ .

The integral terms associated with nonresonant case require a bit more work since (3.7) has  $N_i$  and  $n_3$  as its coefficients. For the frequency interval  $[N^{1/2}, 2N^{1/2}]$ , we can divide this interval into  $O((N')^{1/4})$  intervals uniformly, and then by the orthogonality and the pigeon-hole principle, there exists at least one interval form of  $[M, M + N^{1/4}]$  such that

$$\|(P_{\leq M+N^{1/4}} - P_{\leq M})u\|_{Y^{-1/2}} \lesssim N^{-\sigma}.$$

Fix this  $M$ , we can let  $u_{low} = P_{\leq M} u$ ,  $u_{med} = (P_{\leq M+N^{1/4}} - P_{\leq M}) u$  and  $u_{hi} = (1 - P_{\leq M+N^{1/4}}) u$ . Then by Lemma 3.2,  $(\text{remainder terms})_2$  is bounded by  $\mathcal{O}(N^{-\sigma})$  if it has  $u_{med}$  terms. Moreover, as before, the terms consisting of low frequency terms only are not included in  $(\text{remainder terms})_2$ . Hence, we consider terms which have at least one  $u_{hi}$ . The worst case of this situation is  $P_{\leq M} F_{nr}(u_{hi} v_{low} v_{low})$ . As like the boundary terms, we split the solutions into the dyadic pieces,

$$\sum_{N_0, N_1, N_2, N_3} P_{N_0} P_{\leq M} F_{nr}(P_{N_1} u_{hi}, P_{N_2} v_{low}, P_{N_3} v_{low}).$$

Using frequency relation, we have  $|k_1 + k_2 + k_3| = |k_0| \leq M$ ,  $|k_1| \geq M + N^{1/4}$  and  $|k_2|, |k_3| \leq M$ . We thus have  $n_3 \gtrsim N^{1/4}$ , and from Lemma 3.4,

$$\left\| \sum_{N_0, N_1, N_2, N_3} P_{N_0} P_{\leq M} F_{nr}(P_{N_1} u_{hi}, P_{N_2} v_{low}, P_{N_3} v_{low}) \right\|_{Z^{-1/2}} \lesssim N^{-\sigma}. \quad (4.15)$$

Therefore,  $(\text{remainder terms})_2$  is bounded by  $\mathcal{O}_N(1)$ .

### Step 2.

The argument is based on the mean value theorem. Indeed, we use the smooth truncation instead of the sharp truncation to applying the mean value theorem. Our claim is

$$\begin{aligned} & \left\| P_{\leq N^{1/2}} \mathcal{F}^{-1} \left[ \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0}} \frac{(1-b(k_1+k_2))}{k_3} v_{k_1}^b u_{k_2}^b u_{k_3}^b \right] \right\|_{Z^{-1/2}} \\ & \leq C \left( T, \|u_0^b\|_{H_0^{-1/2}}, \|v_0^b\|_{H_0^{-1/2}} \right) \mathcal{O}_N(1). \end{aligned} \quad (4.16)$$

To prove (4.16), we inspect the support of indices. By the sharp truncation  $P_{\leq N^{1/2}}$  and the smooth truncation  $b(k_1+k_2)$ , we have  $|k_1+k_2+k_3| = |k| \lesssim N^{1/2}$ ,  $|k_1+k_2| \gtrsim N$  and then  $|k_3| \gtrsim N$ , and therefore,  $n_1 \gtrsim N$ . If  $(k_1, k_2, k_3)$  is nonresonant, then (4.16) is directly obtained by Lemma 3.4 and the global well-posedness. The remaining case is the resonant case. As mentioned above, we have  $|k_1+k_2| \gtrsim N$  and  $|k_3| \gtrsim N$ , so there are only two cases. That is,  $(k_1, k_2, k_3) = (k, -k_3, k_3)$  or  $(-k_3, k, k_3)$ . For fixed time  $T$ , we have

$$\left| \sum_{k \in \mathbb{Z}^*} \frac{1}{k} v_{-k} u_k \right| \lesssim \|v\|_{H_0^{-1/2}} \|u\|_{H_0^{-1/2}} \lesssim 1$$

by the Cauchy-Schwarz inequality and the global well-posedness, and then

$$\frac{1}{2} \left| \sum_{k \in \mathbb{Z}^*} \frac{v_{-k} u_k - v_k u_{-k}}{k} \right| = \left| \sum_{k \in \mathbb{Z}^*} \frac{1}{k} v_{-k} u_k \right| \lesssim 1.$$

Hence, we can let  $e_k$  such that

$$e_k = \frac{1}{k} v_{-k} u_k - \frac{1}{k} v_k u_{-k} \text{ and } \sum_{k \gtrsim N} e_k = \mathcal{O}_N(1). \quad (4.17)$$

We can now rewrite a piece of the resonant case as follows,

$$\begin{aligned}
& \sum_{|k_3| \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \frac{1 - b(k - k_3)}{k_3} v_k^b u_{-k_3}^b u_{k_3}^b \\
& + \sum_{|k_3| \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \frac{1 - b(k - k_3)}{k_3} v_{-k_3}^b u_k^b u_{k_3}^b \\
& = \sum_{k_3 \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \frac{b(k + k_3) - b(k - k_3)}{k_3} v_k^b u_{-k_3}^b u_{k_3}^b \\
& + \sum_{k_3 \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \left[ \frac{1 - b(k - k_3)}{k_3} v_{-k_3}^b u_k^b u_{k_3}^b - \frac{1 - b(k + k_3)}{k_3} v_{k_3}^b u_k^b u_{-k_3}^b \right] \\
& = \sum_{k_3 \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \frac{b(k + k_3) - b(k - k_3)}{k_3} v_k^b u_{-k_3}^b u_{k_3}^b \\
& + \sum_{k_3 \gtrsim N} 1_{[-N^{1/2}, N^{1/2}]}(k) \left[ \frac{b(k + k_3) - b(k - k_3)}{k_3} v_{k_3}^b u_k^b u_{-k_3}^b + (1 - b(k - k_3)) e_{k_3} u_k^b \right].
\end{aligned}$$

By the fact that  $b(k)$  is even and the mean value theorem, we have

$$|b(k + k_3) - b(k - k_3)| = |b(k + k_3) - b(k_3 - k)| = \mathcal{O}\left(\frac{|k|}{N}\right) = \mathcal{O}(N^{-\sigma}), \quad (4.18)$$

for  $|k| \lesssim N^{1/2}$ . In addition, we have

$$\left| \sum_{|k_3| \gtrsim N} \frac{1}{k_3} u_{-k_3} v_{k_3} \right| \leq \|u\|_{H_0^{-1/2}} \|v\|_{H_0^{-1/2}}, \quad (4.19)$$

and

$$\|u\|_{L_t^\infty H_x^{-1/2}} \lesssim \|u\|_{Y^{-1/2}}, \quad (4.20)$$

by the Cauchy-Schwarz inequality and the global well-posedness. Thus, (4.16) is proved by (4.17)-(4.20). From (3.1), we conclude the  $Z^{-1/2}$ -term of (4.10) is  $\mathcal{O}_N(1)$ .

### Step 3.

To complete the proof of Theorem 2.9, we need to use a rescaling argument. Our claim is to show that (2.3) is true on the time interval  $[0, T]$ . In fact, this claim is equivalent to show that the  $\alpha$ -scaled problem with solution

$$u_\alpha(x, t) = \alpha^{-2} u\left(\frac{x}{\alpha}, \frac{t}{\alpha^3}\right) \quad (4.21)$$

is true on an interval  $[0, \alpha^3 T]$  and on domain  $\alpha \mathbb{T}$ . Roughly speaking, if we can show that  $\|P_{\leq N^{1/2}} u_\alpha\|_{Y^{-1/2}}$ ,  $\|P_{\leq N^{1/2}} v_\alpha\|_{Y^{-1/2}}$ ,  $\|P_{\leq N^{1/2}} u_\alpha^b\|_{Y^{-1/2}}$  and  $\|P_{\leq N^{1/2}} v_\alpha^b\|_{Y^{-1/2}}$  are sufficiently small in  $\alpha \mathbb{T}$ , then we are done by putting all previous step together.

Although the implicit constants depend on  $\alpha$ , we can obtain (4.9) and (4.10) on  $\alpha\mathbb{T}$  by Remark 3.6. More precisely, by Step 1, Step 2 and Remark 3.6, we have

$$\begin{aligned}
 & \left\| P_{\leq N^{1/2}} \left( u_\alpha - u_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \\
 & \lesssim \alpha^{0+} \left[ \left\| P_{\leq N^{1/2}} v_\alpha \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} \left( v_\alpha - v_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \right. \\
 & + \left\| P_{\leq N^{1/2}} v_\alpha^b \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} \left( v_\alpha - v_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \\
 & + \left\| P_{\leq N^{1/2}} u_\alpha \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} v_\alpha \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} \left( v_\alpha - v_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \\
 & + \left\| P_{\leq N^{1/2}} u_\alpha \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} v_\alpha^b \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} \left( v_\alpha - v_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \\
 & + \left. \left\| P_{\leq N^{1/2}} v_\alpha^b \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} v_\alpha^b \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \left\| P_{\leq N^{1/2}} \left( u_\alpha - u_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \right] \\
 & + \mathcal{O}_N(1).
 \end{aligned} \tag{4.22}$$

Similarly, we also have the estimate with respect to  $\left\| P_{\leq N^{1/2}} \left( v_\alpha - v_\alpha^b \right) \right\|_{Y^{-1/2}(\alpha\mathbb{T})}$ . Hence, if we can prove

$$\left\| P_{\leq N^{1/2}} u_\alpha^{(b)}(t, x) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} + \left\| P_{\leq N^{1/2}} v_\alpha^{(b)}(t, x) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \ll 1, \tag{4.23}$$

for  $N > N_0(T, \varepsilon, \|u_0\|_{H_0^{-1/2}(\mathbb{T})}, \|v_0\|_{H_0^{-1/2}(\mathbb{T})})$ , then all terms of the right hand side of (4.22) except for  $\mathcal{O}_N(1)$  are absorbed in the left hand side. Now we show that (4.23). By the global bound of solutions for  $t \in [0, T]$ , and scaling back from (4.21),

$$\left\| P_{\leq N^{1/2}} u_\alpha(t, x) \right\|_{Y^{-1/2}(\alpha\mathbb{T})} \lesssim \left\| P_{\leq N^{1/2}} u_{\alpha,0}(x) \right\|_{H_0^{-1/2}(\alpha\mathbb{T})} = \alpha^{-1} \left\| P_{\leq \alpha N^{1/2}} u_0 \right\|_{H_0^{-1/2}(\mathbb{T})}.$$

We first choose  $\alpha$  sufficiently large such that terms involving the difference are absorbed to the left hand side. In the estimates of remainder terms in the previous step,  $\mathcal{O}_N(1)$  depends on  $\alpha$ , too. But after fixing  $\alpha$ , we choose  $N$  sufficiently large so that  $\mathcal{O}_N(1)$  to be small. We handle  $v_\alpha$ ,  $u_\alpha^b$  and  $v_\alpha^b$  similarly. Consequently, we finish the proof of Theorem 2.9 due to  $Y^s \subset C_t H^s$ .

#### 4.3. Proof of Theorem 2.8.

The argument is highly similar to [6] to prove Theorem 2.8. We mainly prove the following local-in time estimate.

**Lemma 4.1.** *Let  $N' \geq 1$ ,  $(u_0, v_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$  and  $(u'_0, v'_0) \in H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}}$  such that  $P_{\leq N'}(u_0, v_0) = P_{\leq N'}(u'_0, v'_0)$ . If  $T'$  is sufficiently small depending on  $\|u_0\|_{H_0^{-1/2}}$ ,  $\|u'_0\|_{H_0^{-1/2}}$ ,*

$\|v_0\|_{H_0^{-1/2}}$ , and  $\|v'_0\|_{H_0^{-1/2}}$ , then we have

$$\begin{aligned} & \sup_{|t| \leq T'} \left\| P_{\leq N' - (N')^{1/2}} (S_{CKdV}(t) u_0 - S_{CKdV}(t) u'_0) \right\|_{H_0^{-1/2}} \\ & + \sup_{|t| \leq T'} \left\| P_{\leq N' - (N')^{1/2}} (S_{CKdV}(t) v_0 - S_{CKdV}(t) v'_0) \right\|_{H_0^{-1/2}} \\ & \leq C \left( \|u_0\|_{H_0^{-1/2}}, \|u'_0\|_{H_0^{-1/2}}, \|v_0\|_{H_0^{-1/2}}, \|v'_0\|_{H_0^{-1/2}} \right) \mathcal{O}_{N'}(1). \end{aligned}$$

Theorem 2.8 can be proved by using Lemma 4.1. Roughly speaking, we divide the given time interval  $[-T, T]$  into intervals which has length  $|T'|$ , and use repeatedly Lemma 4.1. For this argument, we refer to Section 5 in [6] for details.

*Proof of Lemma 4.1.* We only consider the difference between  $u$  and  $u'$  as  $v$  and  $v'$  case is handled in the same way. From the local well-posedness for (CKdV), we have the local estimates

$$\|u\|_{Y^{-1/2}} + \|u'\|_{Y^{-1/2}} \lesssim C \text{ and } \|v\|_{Y^{-1/2}} + \|v'\|_{Y^{-1/2}} \lesssim C, \quad (4.24)$$

by choosing the sufficiently small time  $T'$  depending on the  $H_0^{-1/2}$ -norms of  $u_0$ ,  $u'_0$ ,  $v_0$  and  $v'_0$ . We apply  $P_{\leq M}$  in (4.1) to get,

$$\begin{aligned} \partial_t 1_{[-M, M]}(k) \mathbf{u}_k &= \partial_t \left\{ \frac{1_{[-M, M]}(k)}{6} \sum_{k_1 + k_2 = k} \frac{e^{-i\phi(k)t}}{k_1 k_2} \mathbf{v}_{k_1} \mathbf{v}_{k_2} \right\} \\ &+ 2i \frac{1_{[-M, M]}(k)}{6} \sum_{\substack{k_1 + k_2 + k_3 = k \\ k_1 + k_2 \neq 0}} 1_{[-M, M]}(k_1 + k_2) \frac{e^{-i\Phi(k)t}}{k_3} \mathbf{u}_{k_1} \mathbf{v}_{k_2} \mathbf{v}_{k_3}. \end{aligned} \quad (4.25)$$

Taking linear propagator back, we can rewrite the right hand side of (4.25) as follows:

$$\begin{aligned} & \partial_t \mathcal{F}_x [P_{\leq M} B'_2(\mathbf{v}, \mathbf{v})] + \mathcal{F}_x [P_{\leq M} N'_3(\mathbf{u}, \mathbf{v}, \mathbf{v})] \\ &= \partial_t \left\{ \frac{1_{[-M, M]}(k)}{6} \sum_{k_1 + k_2 = k} \frac{e^{-i\phi(k)t}}{k_1 k_2} e^{-it(k_1^3 + k_2^3)} v_{k_1} v_{k_2} \right\} \\ &+ 2i \frac{1_{[-M, M]}(k)}{6} \sum_{\substack{k_1 + k_2 + k_3 = k \\ k_1 + k_2 \neq 0}} 1_{[-M, M]}(k_1 + k_2) \frac{e^{-i\Phi(k)t}}{k_3} e^{-it(k_1^3 + k_2^3 + k_3^3)} u_{k_1} v_{k_2} v_{k_3} \\ &=: \partial_t \mathcal{F}_x [P_{\leq M} B'_2(v, v)] + \mathcal{F}_x [P_{\leq M} N'_3(u, v, v)]. \end{aligned}$$

The constant  $M$  will be the low frequency cut-off and will be chosen later. In order to show  $\|P_{N' - (N')^{1/2}} u(t) - u'(t)\|_{Y^{-1/2}}$  to be small for a short time  $T'$ , we analyse the nonlinear terms and show that the contribution from high frequency pieces is small and so regarded as remainder terms in  $\mathcal{O}_{N'}(1)$ . In addition, for the contribution from all low frequency pieces, we use the local stability theory.



We first consider the trilinear term  $N'_3(u, v, v)$ . This part is also similar to Step 1 of Subsection 4.2. In order to control  $N'_3(u, v, v)$ , we define the  $(error\ terms)_3$  which has the  $Z^{-\frac{1}{2}}$ -norm of  $\mathcal{O}_{N'}(1)$ . The solutions  $u, u', v$  and  $v'$  are decomposed into the three pieces using the following argument. By the global well-posedness and the pigeon-hole principle, we may find an interval  $[M, M + (N')^{1/4}] \subseteq [N' - (N')^{\frac{1}{2}}, N']^3$  such that

$$\begin{aligned} & \left\| (P_{\leq M+(N')^{1/4}} - P_{\leq M})u \right\|_{Y^{-1/2}} + \left\| (P_{\leq M+(N')^{1/4}} - P_{\leq M})u' \right\|_{Y^{-1/2}} \\ & + \left\| (P_{\leq M+(N')^{1/4}} - P_{\leq M})v \right\|_{Y^{-1/2}} + \left\| (P_{\leq M+(N')^{1/4}} - P_{\leq M})v' \right\|_{Y^{-1/2}} \lesssim (N')^{-\sigma}. \end{aligned} \quad (4.26)$$

We fix such  $M$  with  $N' - N'^{1/2} \leq M \leq N'$  and decompose  $u$  as

$$u = u_{low} + u_{med} + u_{hi},$$

where

$$u_{low} := P_{\leq M}u, \quad u_{med} := (P_{\leq M+(N')^{1/4}} - P_{\leq M})u, \quad u_{hi} := (1 - P_{\leq M+(N')^{1/4}})u.$$

From (4.24) and (4.26), we have

$$\|u_{low}\|_{Y^{-1/2}}, \|u_{hi}\|_{Y^{-1/2}} \leq C \quad \text{and} \quad \|u_{med}\|_{Y^{-1/2}} \lesssim (N')^{-\sigma}. \quad (4.27)$$

We also do the same decomposition for  $u', v$  and  $v'$ , and obtain analogous estimates like (4.27). Moreover, we denote

$$P_{\leq M}N'_3(u, v, v) = P_{\leq M}N'_3(u_{low}, v_{low}, v_{low}) + (\text{remainder terms})_3.$$

First of all, from Lemma 3.3 and 3.4, any term in  $(\text{remainder terms})_3$  involving  $u_{med}, u'_{med}, v_{med}$  or  $v'_{med}$  is  $\mathcal{O}\left((N')^{-\sigma}\right)$  in  $(error\ terms)_3$ . We now consider terms which involve in  $v_{hi}$ . As before,  $(\text{remainder terms})_3$  is split into the resonant case and the nonresonant case. From (4.13), the typical term of resonant case is  $P_{\leq M}N'_3(u_{low}, v_{hi}, v_{hi})$  and therefore, we have

$$\|P_{\leq M}N'_3(u_{low}, v_{hi}, v_{hi})\|_{Z^{-1/2}} \sim \mathcal{O}_{N'}(1),$$

by (4.14). For the nonresonant case, we estimate as Step 1 of Subsection 4.2. Since we have  $n_3 \gtrsim (N')^{1/4}$ , Lemma 3.4, and the estimate of (4.15) in Subsection 4.2, the nonresonant case of  $(\text{remainder terms})_3$  is bounded by  $\mathcal{O}\left((N')^{-\sigma}\right)$ . In other words,  $P_{\leq M}N'_3(u, v, v)$  can be written  $P_{\leq M}N'_3(u_{low}, v_{low}, v_{low}) + (error\ terms)_3$  with  $\|(error\ terms)_3\|_{Z^{-1/2}} = \mathcal{O}_{N'}(1)$ . Due to  $P_{\leq M}e^{\pm t\partial_x^3} = e^{\pm t\partial_x^3}P_{\leq M}$ , we have

$$P_{\leq M}N'_3(u, v, v) = P_{\leq M}N'_3(\mathbf{u}_{low}, \mathbf{v}_{low}, \mathbf{v}_{low}) + (error\ terms)_3,$$

where  $\mathbf{u}_{low} = P_{\leq M}\mathbf{u}$  and  $\mathbf{v}_{low} = P_{\leq M}\mathbf{v}$ .

The bilinear term  $B'_2(v, v)$  can be analysed in a similar way. Let  $(error\ terms)_4$  be term which has the  $Y^{-\frac{1}{2}}$ -norm of  $\mathcal{O}_{N'}(1)$ . We choose the same constant  $M$  in  $N'_3(u, v, v)$ -term case and split the solution  $v$  and  $v'$  into as follows,

$$v = v_{low} + v_{hi},$$

---

<sup>3</sup> The constant  $M$  shall be different from that in Subsection 4.2.

where

$$v_{low} = P_{\leq M} v \quad \text{and} \quad v_{hi} = (1 - P_{\leq M}) v.$$

We denote

$$P_{\leq M} B'_2(v, v) = P_{\leq M} B'_2(v_{low}, v_{low}) + (\text{remainder terms})_4. \quad (4.28)$$

From (4.28),  $(\text{remainder terms})_4$  has  $v_{hi}$  term only. Hence,  $(\text{remainder terms})_4$  is bounded by  $\mathcal{O}_{N'}(1)$  from (4.11) and  $M \in [N' - (N')^{\frac{1}{2}}, N']$ . Therefore, the bilinear term  $B'_2(v, v)$  can be written  $B'_2(v_{low}, v_{low}) + (\text{error terms})_4$ , and so

$$B'_2(v, v) = B'_2(\mathbf{v}_{low}, \mathbf{v}_{low}) + (\text{error terms})_4,$$

by  $P_{\leq M} e^{\pm t \partial_x^3} = e^{\pm t \partial_x^3} P_{\leq M}$ .

Consequently,  $\mathbf{u}_{low}$  obeys the equation,

$$\partial_t \mathbf{u}_{low} = \partial_t P_{\leq M} B'_2(\mathbf{v}_{low}, \mathbf{v}_{low}) + P_{\leq M} N'_3(\mathbf{u}_{low}, \mathbf{v}_{low}, \mathbf{v}_{low}) + \sum_{i=3,4} (\text{error terms})_i. \quad (4.29)$$

In the same manner, the function  $\mathbf{u}'_{low}$  obeys the equation

$$\partial_t \mathbf{u}'_{low} = \partial_t P_{\leq M} B'_2(\mathbf{v}'_{low}, \mathbf{v}'_{low}) + P_{\leq M} N'_3(\mathbf{u}'_{low}, \mathbf{v}'_{low}, \mathbf{v}'_{low}) + \sum_{i=3,4} (\text{error terms})_i. \quad (4.30)$$

From the local well-posedness for (4.29) or (4.30), transforming back,  $u_{low}(0) = u'_{low}(0)$ , and the fact that by the rescaling argument as in Subsection 4.2, we may assume that the initial data are small in  $Y^{-1/2}$ , we have

$$\|u_{low} - u'_{low}\|_{Y^{-1/2}} \lesssim \mathcal{O}_{N'}(1).$$

We also get

$$\|v_{low} - v'_{low}\|_{Y^{-1/2}} \lesssim \mathcal{O}_{N'}(1),$$

by the similar argument, and we thus finish the proof by  $Y^s \subset C_t H^s$ .  $\square$

**Remark 4.2.** *Although we provide the proof for (1.3), the same proof works for a more general case (1.2) if there is a global control of solutions on  $C_t H_0^{-\frac{1}{2}}([0, T] \times \mathbb{T})$ .*

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